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Completeness Theorems for Poincare Series in One Variable

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A completeness theorem for Poincaré series asserts that the space of theta-series of some specified kind coincides with or is dense in some space of automorphic forms. Such theorems occur in the well-known investigations by H. Petersson on finitely generated Fuchsian groups with the real axis as limit circle.[2] In connection with the problem of moduli, I had occasion to consider the same question for arbitrary Fuchsian groups and also for discrete groups of conformal self-mappings of an arbitrary simply connected domain. The results, though not the application to the theory of moduli which provided the motivation, are presented here. The last section is an exposition of some of Petersson's results, with modifications and simplifications resulting from using the unit disc instead of the upper half-plane. The methods are elementary. We use only the simplest concepts from the theory of linear spaces, the scalar product introduced by Petersson, a device utilized in a recent thesis by E. Rodlitz[3], and some simple potential-theoretical estimates.

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[1] The author is a John Simon Guggenheim Memorial Fellow and holds a Fulbright award.

[2] Cf., for instance, Math. Z. vol. 52 (1949). pp. 32-59, and the the references given there.

[3] New York University, 1960, to appear in Comm. Pure Appl. Math.

§1. Automorphic forms.

Let there be given, once and for all, a plane domain D. All functions considered will be assumed to be complex-valued, measurable and defined in D; partial derivatives will be understood in the L_1 sense; functions coinciding a.e. will be identified. We use the usual norms

$$\|F\|_p = \left\{ \iint_D |F(z)|^p \, dxdy \right\}^{1/p}, \qquad 1 \leq p < \infty ,$$

$$\|F\|_\infty = \text{essential supremum } |F(z)| ,$$

and the scalar product

$$(F_1, F_2) = \iint_D F_1(z) \overline{F_2(z)} \, dxdy .$$

Let there be given, once and for all, a discrete group G of conformal self-mappings $z \to A(z)$ of D. If $\Delta \subset D$ is an open set, we say that Δ is an allowable fundamental domain if: the relative boundary $\dot{\Delta}$ of Δ has two-dimensional measure 0, no two points of Δ are equivalent under G, and every point of D is G-equivalent to some point of $\Delta \cup \dot{\Delta}$.

We call $\lambda(z)|dz|$ an invariant line element if $\lambda(z)$ is positive and continuous, and

$$(1) \qquad \lambda(A(z))|A'(z)| = \lambda(z) , \quad z \in D , \quad A \in G .$$

If D is the unit disc, the Poincaré line element is defined by

$$(2) \qquad \lambda(z) = (1 - |z|^2)^{-1} .$$

If D is simply connected and has more than one boundary point, the Poincaré line element is defined by

$$\lambda(z) = |\psi'(z)|(1 - |\psi(z)|^2)^{-1} \tag{2'}$$

where $z \longrightarrow \psi(z)$ is any conformal mapping of D onto the unit disc. The Poincaré metric is invariant; in fact, (1) holds for all conformal self-mappings of D.

Let there be given, once and for all, a real number s and a system of holomorphic functions $\rho_A(z)$, $A \in G$, $z \in D$, satisfying the relations

$$|\rho_A(z)| = |A'(z)|^{-s} , \tag{3}$$

$$\rho_{AB}(z) = \rho_A(B(z))\rho_B(z) . \tag{4}$$

Every measurable solution of the functional equation

$$f(A(z)) = \rho_A(z)f(z) , \quad A \in G, \quad z \in D , \tag{5}$$

is called an automorphic form (to the group G, of dimension $(-2s)$, and with automorphy factors ρ_A). We denote by $\underline{A}$ the complex vector space of automorphic forms and by $\underline{H} \subset \underline{A}$ the subspace of holomorphic automorphic forms.

Having chosen an invariant line element $\lambda(z)|dz|$ we set, for $f \in \underline{A}$,

$$\|f\|_p^{\#} = \left\{ \iint_{\triangle} \lambda(z)^{2-ps}|f(z)|^p \, dxdy \right\}^{1/p} , \quad 1 \leq p < \infty ,$$

$$\|f\|_{\infty}^{\#} = \text{essential supremum } \lambda(z)^{-s}|f(z)| ,$$

and for $f_1, f_2 \in \underline{A}$,

$$(f_1,f_2)^{\#} = \iint_{\triangle} \lambda(z)^{2-2s} f_1(z)\overline{f_2(z)} \, dxdy ,$$

provided the integral exists. Here $\triangle$ is any allowable fundamental domain.

Let $\underline{A}_p$ denote the subspace of $\underline{A}$ defined by $\|f\|_p^{\#} < \infty$, and set $\underline{H}_p = \underline{H} \cap \underline{A}_p$, One sees at once that $\underline{A}_p$ and $\underline{H}_p$ are Banach spaces. The scalar product $(\, , \,)^{\#}$ defines a canonical identification of $\underline{A}_{p/(p-1)}$ with $\underline{A}_p^*$ for $1 \leq p < \infty$ and of $\underline{A}_1$ with a subspace of $\underline{A}_\infty^*$.

§2. Poincaré series.

To every function $F(z)$, $z \in D$, there is associated a Poincaré theta-series

$$(6) \qquad \sum_{A \in G} F(A(z))\rho_A(z)^{-1} .$$

If this series converges absolutely in the L_1 sense over every compact $K \subset D$, we denote the sum by ΘF; it is clear that $\Theta F \in \underline{A}$. Conversely, every $f \in \underline{A}$ is of the form ΘF; it suffices, for instance, to set $F(z) = k(z)f(z)$ where k is the characteristic function of a measurable set containing exactly one representative of each G-equivalence class. On the other hand, the problem of representing or approximating an $f \in \underline{H}$ by expressions of the form ΘF, $F(z)$ a polynomial, is not trivial. As a preparation for treating it we prove

Theorem 1. If $\|\lambda^{2-s}F\|_1 < \infty$, then (6) converges absolutely in the L_1 sense over every compact $K \subset D$,

$$(7) \qquad \|\Theta F\|_1^{\#} \leq \|\lambda^{2-s} F\|_1 ,$$

and

$$(8) \qquad (\Theta F, g)^{\#} = (\lambda^{2-s}F, g)$$

for all $g \in \underline{A}_{\infty}$.

Proof. Let $\triangle$ be an allowable fundamental domain. We call a set $K \subset D$ small if $K \subset \triangle \cup \triangle$. If $\hat{K} \subset D$ is any compact set, then there exists a small compact K and finitely many elements $A_1, \ldots, A_n$ of G such that $\hat{K} \subset A_1(K) \cup \ldots \cup A_n(K)$. Hence the convergence statement of the theorem will be proved if we show that

$$(9) \qquad \sum \iint_K |F(A(z))|\,|A'(z)|^s \ dxdy < \infty$$

for every small compact $K \subset D$. This follows by noting that for $z \in K$ and $A \in G$ the quantity $\lambda(A(z))|A'(z)|$ remains between two positive bounds, namely the minimum and maximum of $\lambda(z)$ in K, so that the sum in (9) does not exceed a constant multiple of

$$\sum \iint_K |F(A(z))|\lambda)A(z))^{2-s}|A'(z)|^2 \ dxdy =$$

$$\sum \iint_{A(K)} |F(z)|\lambda(z)^{2-s} \ dxdy \leq \|\lambda^{2-s}F\|_1 \ .$$

Note, let us call a $g \in \underline{A}_{\infty}$ special if there exists an allowable fundamental domain $\triangle$, a compact $K \subset \triangle$ and an $\alpha > 0$, such that, in $\triangle$, $g = 0$ at points outside K and also at points where $|\lambda^{2-s}\Theta F| > \alpha$. For a special g, $(\Theta F, g)^{\#}$ exists and (8) can be verified by the following legitimate computation:

$$(\theta F,g)^{\#} = \sum \iint_{\triangle} \lambda(z)^{2-2s} F(A(z))\, \rho_A(z)^{-1}\, \overline{g(z)}\, dxdy$$

$$= \sum \iint_{\triangle} \lambda(A(z))^{2-2s} F(A(z)) \overline{g(A(z))} |A'(z)|^2\, dxdy$$

$$= \sum \iint_{A(\triangle)} \lambda(z)^{2-2s} F(z) \overline{g(z)}\, dxdy = (\lambda^{2-2s}F,g) .$$

We conclude that

$$|(\theta F,g)^{\#}| \leq \|\lambda^{-s}g\|_{\infty} \|\lambda^{2-s}F\|_1 = \|g\|_{\infty}^{\#} \|\lambda^{2-s}F\|_1$$

for special g. This implies (7). Now an easy limiting argument shows that (8) holds also for all $g \in \underline{A}_{\infty}$.

Corollary. If F(z) is holomorphic and $\|\lambda^{2-s}F\|_1 < \infty$, then the series (6) converges normally and $\theta F \in \underline{H}_1$.

The proof is clear.

Remark. Statement and proof of Theorem 1 are at once extendable to non-analytic ρ_A, to non-conformal A, and to higher dimensions.

§3. A completeness theorem for general Fuchsian groups.

If D is the unit disc, λ is defined by (2), s > 1, and F(z) is a polynomial, then the corollary to Theorem 1 implies that $\theta F \in \underline{H}_1$. We shall prove a partial converse of this statement.

Theorem 2. Assume that D is the unit disc, $\lambda|dz|$ the Poincaré line element, and s > 1. Then the space of Poincaré series θF, F(z) a polynomial, is dense in H_1.

The proof is based on the following

Lemma. *Assume that*

$$|\sigma(z)| \leq c(1-|z|)^{\alpha}, \quad \alpha > -1, \tag{10}$$

and

$$\iint_{|z|<1} F(z)\sigma(z)\, dxdy = 0 \text{ for all polynomials } F(z). \tag{11}$$

Set

$$h(z) = -\frac{1}{\pi} \iint_{|\zeta|<1} \frac{\sigma(\zeta)d\xi d\eta}{\zeta - z}. \tag{12}$$

Then (i) h(z) *is continuous for all* z, *has* L_1 *derivatives, and satisfies for* $|z| < 1$ *the equation*

$$h_{\bar{z}} = \sigma. \tag{13}$$

Also, (ii)

$$h = 0 \text{ on } |z| = 1, \tag{14}$$

and (iii)

$$|h(z)| \leq c_1(\alpha)c(1-|z|)^{1+\alpha}[1-\log(1-|z|)], \quad |z| < 1, \tag{15}$$

where the logarithmic term may be omitted if α *is not an integer.*

Proof of the Lemma. a) If $\alpha \geq 0$, statement (i) is well known; it is also known that in this case h has a modulus of continuity of the form $c_1(0)c\varepsilon(1-\log\varepsilon)$. If $-1 < \alpha < 0$, one verifies easily that (i) is still valid and that h has a modulus of continuity of the form $c_1(\alpha)c\varepsilon^{1+\alpha}$.

b) Condition (11) implies easily that h = 0 in $|z| > 1$; (ii) follows by the continuity of h.

c) We set $\alpha = k-1+\delta$, $k = 0,1,\ldots$, $0 < \delta \leq 1$, and prove (iii) by induction on k. In view of a) and b) the statement is true for k = 0. Assume it to be true for k = n, and consider the case

$k = n+1$. Set, for some fixed real t,

$$\sigma_1(z) = \frac{\sigma(z)}{z - e^{it}}, \qquad h_1(z) = -\frac{1}{\pi} \iint\limits_{|\zeta|<1} \frac{\sigma_1(\zeta)\, d\xi\, d\eta}{\zeta - z}.$$

Then

$$|\sigma_1(z)| \leq c(1-|z|)^{\alpha-1} = c(1-|z|)^{n-1+\delta}$$

$$\iint\limits_{|z|<1} F(z)\sigma_1(z)\, dxdy = 0 \text{ for all polynomials } F(z),$$

and, by the induction hypothesis,

$$|h_1(z)| \leq c_1(n-1+\delta)c(1-|z|)^{n+\delta}[1-\log(1-|z|)].$$

But

$$h(re^{it}) = h(re^{it}) - h(e^{it}) = (r-1)e^{it}\, h_1(re^{it}),$$

so that (15) holds also for $k = n+1$.

In order to prove Theorem 2 it suffices to show that if

(16) $$g \in \underline{A}_\infty, \quad (\partial F, g)^\# = 0 \text{ for all polynomials } F(z),$$

then

(17) $$(f,g)^\# = 0 \quad \text{for all} \quad f \in \underline{H}_1.$$

Assume (16) and set

(18) $$\sigma(z) = \lambda(z)^{2-2s}\, \overline{g(z)}.$$

By virtue of (2), (8) and (16) σ satisfies the hypotheses of our lemma with

(19) $$c = 2^{s-2}\|g\|_\infty^\#, \qquad \alpha = s-2 > -1.$$

Next, define h by (12). Then

$$h(A(z))\rho_A(z)A'(z) - h(z) = 0 \quad \text{for} \quad A \in G . \tag{20}$$

To verify this, note that the $\bar{z}$-derivative of the left hand side of (20) equals (cf. (13))

$$\lambda(A(z))^{2-2s}\ \overline{g(A(z))}\ \overline{A'(z)}\rho_A(z)A'(z) - \lambda(z)^{2-2s}\ \overline{g(z)} = 0 ,$$

so that this left hand side is holomorphic. But by virtue of (14) it vanishes on $|z| = 1$. It follows from (20) that

$$\text{for} \quad f \in \underline{H} , \quad f(z)h(z)dz \quad \text{is } G\text{-invariant} , \tag{21}$$

that is, that $\psi = fh$ satisfies the functional equation $\psi(A(z))A'(z) = \psi(z)$ for all $A \in G$.

We assume now that

$$G \text{ contains no rotations about the origin} \tag{22}$$

(except, of course, for the identity) and denote by $\triangle$ the set of those $z \in D$ for which $|A(z)| > |z|$ for $A \in G$, $A \neq 1$. Let $\triangle_r$ denote the intersection of $\triangle$ with the disc $|z| < r < 1$, B_r the part of the boundary of $\triangle_r$ lying in $|z| < r$, and C_r the part of the boundary of $\triangle_r$ which lies on $|z| = r$. It is well known that $\triangle$ is an allowable fundamental domain (called the normal fundamental polygon with center at 0). Hence, for $f \in \underline{H}_1$,

$$\begin{aligned}(f,g)^{\#} &= \lim_{r\uparrow 1} \iint_{\triangle_r} \lambda^{2-2s} f\bar{g}\, dxdy = \lim_{r\uparrow 1} \iint_{\triangle_r} (fh)_{\bar{z}}\, dxdy \\ &= -(i/2) \lim_{r\uparrow 1} \int_{B_r} fh\, dz - (i/2) \lim_{r\uparrow 1} \int_{C_r} fh\, dz .\end{aligned} \tag{23}$$

It is well known and easy to check that B_r consists of a finite number of circular arcs $b_1,\dots,b_n$, $b_1',\dots,b_n'$ such that

$$(24)\qquad b_j' = -A_j(b_j)\ , \quad A_j \in G\ , \qquad j = 1,\dots,n\ .$$

In conjunction with (21) this implies that

$$(25)\qquad \int_{B_r} fh\,dz = 0\ .$$

By (15) and (24)

$$h(z) = O(-(1-r)^{s-1}\log(1-r)) \quad \text{for} \quad |z| = r\uparrow 1\ ,$$

and therefore

$$\int_{C_r} fh\,dz = O(-(1-r)^{s-1}\log(1-r)\int_{C_r}|f|\,|dz|)\ , \qquad r\uparrow 1\ .$$

On the other hand, since $f \in \underline{H}_1$,

$$\int_{1/2}^{1}(1-r^2)^{s-2}\int_{C_r}|f|\,|dz|\,dr = \iint_{\Delta-\Delta_{1/2}} \lambda^{2-s}|f|\,dxdy < \infty\ ,$$

so that

$$\int_{1/2}^{1}\left|\int_{C_r} fh\,dz\right|\frac{dr}{(1-r)|\log(1-r)|} < \infty$$

and therefore

$$(26)\qquad \liminf_{r\uparrow 1}\left|\int_{C_r} fh\,dz\right| = 0\ .$$

By (23), (25) and (26), $(f,g)^{\#} = 0$, q.e.d. The additional hypothesis (22) will be removed in the next section.

§4. Extension to discrete groups of conformal self-mappings of an arbitrary simply connected domain.

This extension is an almost immediate consequence of Theorem 2 with the additional hypothesis (22).

Let $\zeta \rightarrow z = \psi(\zeta)$ be a conformal mapping of the unit disc $\hat{D} : |\zeta| < 1$ onto a simply connected domain D and let $\psi'(\zeta)^s$ denote some single-valued branch of this function in $\hat{D}$. For $A \in G$, $\hat{A} = \psi^{-1}A\psi$ is a conformal self-mapping of $\hat{D}$; the group $\hat{G}$ of all such $\hat{A}$ is discrete. We may assume that ψ is chosen so that $\hat{G}$ contains no rotations about the origin. Set

$$\hat{\rho}_{\hat{A}}(\zeta) = \rho_A(\psi(\zeta))\psi'(\hat{A}(\zeta))^s\psi'(\zeta)^{-s} , \quad \hat{A} \in \hat{G} .$$

Then $|\hat{\rho}_{\hat{A}}(\zeta)| = |\hat{A}'(\zeta)|^{-s}$ and $\{\hat{\rho}_{\hat{A}}\}$ is a system of automorphy factors belonging to $\hat{G}$. We denote by $\hat{\underline{H}}_p$ the spaces of holomorphic automorphic forms to the group $\hat{G}$, of dimension $(-2s)$ and with automorphy factors $\hat{\rho}_{\hat{A}}$, the norms being based on the Poincaré line element in $\hat{D}$. We also use the Poincaré line element in D to define the norms in $\underline{H}$.

To every function F(z), $z \in D$, we associate the function

$$\hat{F}(\zeta) = \psi'(\zeta)^s F(\psi(\zeta))$$

and we write $\hat{F} = \Phi F$. One verifies at once that Φ is an isometric and isomorphic mapping of $\underline{H}_p$ onto $\hat{\underline{H}}_p$, $1 \leq p \leq \infty$, and also of the Banach space of holomorphic functions in D with norm $\|\lambda^{2-s}F\|_1$ onto the corresponding Banach space of holomorphic functions in $\hat{D}$.

Denote the Poincaré series formed for the group $\hat{G}$ and the automorphy factors $\hat{\rho}_{\hat{A}}$ by $\hat{\Theta}$. Then

$$\hat{\Theta}\Phi F = \Phi\Theta F .$$

Thus the part of Theorem 2 already proved yields

Theorem 3. *If* D *is simply connected and not the whole plane,* $\lambda|dz|$ *is the Poincaré line element in* D, *and* $s > 1$, *then the space of Poincaré series* ΘF, $F(z)$ *holomorphic and* $\|\lambda^{2-s}F\|_1 < \infty$, *is dense in* $\underline{H}_1$.

Now we can prove Theorem 2 without the additional hypothesis (22). Noting Theorem (3) and (7), we see that all one needs is the following statement: in the Banach space of holomorphic functions in the unit disc D with norm $\|\lambda^{2-s}F\|_1$ the polynomials form a dense set. But this statement is simply Theorem 2 for the case when G contains only the identity. Thus it is already proved in §3.

The usefulness of Theorem 3 is limited by the difficulty of describing explicitly those holomorphic functions for which $\|\lambda^{2-s}F\|_1$ is finite. A more precise statement will be made in the next section under restrictive assumptions on D and G.

§5. A special completeness theorem.

The set D/G can be made, canonically, into a Riemann surface, and this surface is closed if G has a compact fundamental region $\triangle \cup \dot{\triangle}$, as we assume in this section. This assumption implies that

$$\underline{H} = \underline{H}_1 \tag{27}$$

(for every λ) and that

$$\dim \underline{H} < \infty . \tag{28}$$

The latter relation is proved by noting that for f, $g \in H$, $g \not\equiv 0$, the ratio f/g may be considered as a meromorphic function on D/G which is a multiple of a certain divisor determined by g.

If we also assume that D is the unit disc, and $s > 1$, then, by Theorem 2, every $f \in \underline{H}$ is a Poincaré series ΘF, F a polynomial. We shall extend this result to bounded Jordan domains, but only for $s \geq 2$. (The case $1 < s < 2$ could be included under certain smoothness conditions on the boundary of D, but the domains occurring in applications are either discs or bounded by non-differentiable Jordan curves.)

Theorem 4. *Assume that* D *is a bounded simply connected domain, and that its boundary* $\overset{\circ}{D}$ *is a Jordan curve* (*or, more generally, that* $\dot{D}$ *is also the boundary of the complement of the closure of* D). *Assume also that* G *has a compact fundamental region, and that* $s \geq 2$. *Then every element of* $\underline{H}$ *is of the form* ΘF, $F(z)$ *a polynomial.*

Proof. Let $\triangle$ be an allowable fundamental domain. Since

$$\sum \iint_{\triangle} |A'(z)|^2 \, dxdy = \iint_{D} dxdy \, ,$$

the series $\sum |A'(z)|^2$ converges normally, as is well known. This implies, by the way, that (6) converges normally for $s \geq 2$ and $\|F\|_\infty < \infty$. Set

$$\lambda(z) = \left\{ \sum |A'(z)|^2 \right\}^{1/2} . \tag{29}$$

Then $\lambda > 1$, and $\lambda(z)|dz|$ is an invariant line element which we use from now on.

The proof parallels that of Theorem 2. Noting (27) and (28) we see that Theorem 4 will be proved once we derive (17) from (16). We assume (16) and define $\sigma(z)$, $z \in D$, by (18), where λ is given by (29). Then $\|\sigma\|_\infty < \infty$ and σ satisfies (11). It follows that the function

$$h(z) = -\frac{1}{\pi}\iint_D \frac{\sigma(\zeta)\,d\xi d\eta}{\zeta - z}$$

is continuous, has L_1 derivatives, satisfies (13) in D, and vanishes identically in the complement of $D \cup \overset{\circ}{D}$. In view of the hypothesis on $\overset{\circ}{D}$, $h = 0$ on $\overset{\circ}{D}$. As in the proof of Theorem 2, this relation implies (21).

Now, Δ can be chosen so that its boundary $\overset{\circ}{\Delta}$ consists of rectifiable arcs $b_1, \ldots, b_n$, $b_1', \ldots, b_n'$ satisfying (24). For $f \in \underline{H}$ we have by (21)

$$\int_{\overset{\circ}{\Delta}} fh\, dz = 0 ,$$

and therefore

$$(f,g)^{\#} = \iint_{\Delta} \lambda^{2-2s} f\bar{g}\, dxdy = \iint_{\Delta} (fh)_{\bar{z}}\, dxdy = 0 ,$$

q.e.d.

If D is the unit disc, a more precise statement can be made, under weaker conditions on G.

§6. Poincaré series for finitely generated Fuchsian groups with limit circle.

Throughout this, essentially expository, section we make the following hypothesis: G is a finitely generated discrete group of conformal self-mappings of the unit disc D, and the limit points of

G are dense on the unit circumference. This is the case considered by Petersson for D the upper half-plane.

The hypothesis implies that every normal fundamental polygon $\underline{\triangle} \cup \underline{\triangle}$ of G is either compact, if G contains no parabolic elements, or has a finite number of vertices, but no edges, on the unit circumference. Each of these vertices is a cusp, i.e. a fixed point $e^{i\phi}$ of a parabolic element $A \in G$. An $f \in \underline{H}$ is called a cusp form if, for every cusp $e^{i\phi}$ of $\underline{\triangle}$, $f(z) \to 0$ as $z \to e^{i\phi}$ in $\underline{\triangle}$. We denote the space of cusp forms by $\underline{H}_+$; if there are no cusps, then $\underline{H} = \underline{H}_+$. In all cases,

$$(30) \qquad N = \dim \underline{H}_+ < \infty$$

and, for $N > 0$,

$$(31) \qquad M = \sup \operatorname{ord}_z f < \infty \qquad (z \in D,\ f \in \underline{H}_+) \quad .$$

The latter relation means that there exists a $z_0 \in D$ and an $f \in \underline{H}_+$ with $\operatorname{ord}_{z_0} f = M$, i.e. with

$$f(z) = a(z - z_0)^M + O(|z - z_0|^{M+1}), \quad a \neq 0,$$

and that $\operatorname{ord}_z f \leq M$ for all $z \in D$ and all $f \in \underline{H}_+$, $f \not\equiv 0$.

Relations (30) and (31) are proved by observing that D/G is either a closed Riemann surface S, or obtained from a closed Riemann surface S by removing a finite number of points. (The genus of S is, by definition, the genus of G.) If f and $g \not\equiv 0$ are cusp forms, then f/g may be considered as a meromorphic function on S which is a multiple of a certain divisor determined by g.

At every point z of D one defines N positive integers $\mu_1 < \mu_2 < \dots < \mu_N$ (the 'Weierstrass gap sequence of $\underline{H}$ at z') by setting $\mu_1 - 1 = \min \operatorname{ord}_z f$, $f \in \underline{H}_+$, and $\mu_{j+1} - 1 = \min \operatorname{ord}_z f$ for $f \in \underline{H}_+$ and $\operatorname{ord}_z f > \mu_j - 1$. The point z is called a Weierstrass point unless $\mu_j = j$, $j = 1, 2, \dots, N$. It is clear that the μ_j are well defined, if $N > 0$, that $\mu_N \leq M + 1$, and that the set of Weierstrass points is discrete. If G contains no parabolic or elliptic transformations and $\rho_A(z) = A'(z)$, then the Weierstrass gap sequence is the classical one. The actual computation of N and M, and of upper and lower bounds for the number of non-equivalent Weierstrass points, discussed by Petersson, need not concern us here.

If we use the Poincaré line element, then

$$\underline{H}_+ = \underline{H}_p, \qquad 1 \leq p \leq \infty$$

as can be verified by simple estimates.

Lemma. If $s > 1$, $F(z) = \sum a_n z^n$ is a polynomial, and $g(z) = \sum b_n z^n$ a cusp form, then

$$(\theta F, g)^{\#} = \sum \gamma_n a_n \bar{b}_n$$

where

$$\gamma_n = 2\pi \int_0^1 (1-r^2)^{2s-2} r^{2n+1} \, dr .$$

This corresponds to one of Petersson's 'fundamental identities'. The proof consists of substituting the expressions for F and g into the right hand side of (8).

Corollary. *If* $s > 1$, $F = \sum \alpha_n z^n$ *is a polynomial, and* $\Theta F = \sum a_n z^n$, *then*

$$\sum \gamma_n \alpha_n \bar{a}_n \geq 0 ,$$

and the equality sign holds if and only if $\Theta F \equiv 0$.

Proof. The sum equals $(\Theta F, \Theta F)^{\#}$.

Theorem 5. *If* $s > 1$ *and* $\mu_1, \ldots, \mu_N$ *is the Weierstrass gap sequence of* H_+ *at* $z = 0$, *then every cusp form is representable, uniquely, as* ΘF, *with*

$$F(z) = \sum_{j=1}^{N} \alpha_j z^{\mu_j - 1} .$$

Proof. It suffices to show that if F has the indicated form and $\Theta F \equiv 0$, then $\alpha_1 = \ldots = \alpha_N = 0$. But there exists cusp forms f_j, $j = 1, \ldots, N$ with $f_j = z^{\mu_j - 1} + O(|z|^{\mu_j})$. If $\Theta F \equiv 0$, then $0 = (\Theta F, f_N)^{\#} = \gamma_N \alpha_N$, $0 = (\Theta F, f_{N-1})^{\#} = \gamma_{N-1} \alpha_{N-1}$, etc.

Corollary. *If* $s > 1$, *then every cusp form can be written as* ΘF, *where* F *is a polynomial of degree at most* M. *If* $z = 0$ *is not a Weierstrass point of* H_+, *then every cusp form can be written* uniquely *as* ΘF, *where* F *is a polynomial of degree at most* N-1.

We conclude by deriving a rather special consequence of Theorem 5.

Theorem 6. *Assume that* G *is of positive genus and contains no rotations about the origin.* *Then*

$$\sum_{A \in G} A'(0)^k > 0 \qquad \text{for } k = 2,3,\ldots \,.$$

Proof. For a fixed integer $k \geq 2$ set $s = k$ and $\rho_A(z) = A'(z)^{-k}$. The sum considered is, by the lemma, equal to $(\Theta 1, \Theta 1)^{\#}$. It is real and nonnegative and can vanish only if $\Theta 1 \equiv 0$. By Theorem 5, however, $\Theta 1 \not\equiv 0$ if $\mu_1 = 1$, i.e., if there exists an $f \in \underline{H}_+$ with $f(0) \neq 0$. In the present circumstances such an f is given by the k-th power of an Abelian differential of the first kind on S which does not vanish at the point covered by $z = 0$.

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